The NLO Inclusive Forward Hadron Production in pA Collisions

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- G. Chirilli, BX and F. Yuan, Phys. Rev. Lett 108, 122301 (2012).
- G. Chirilli, BX and F. Yuan, arXiv:1203.6139.

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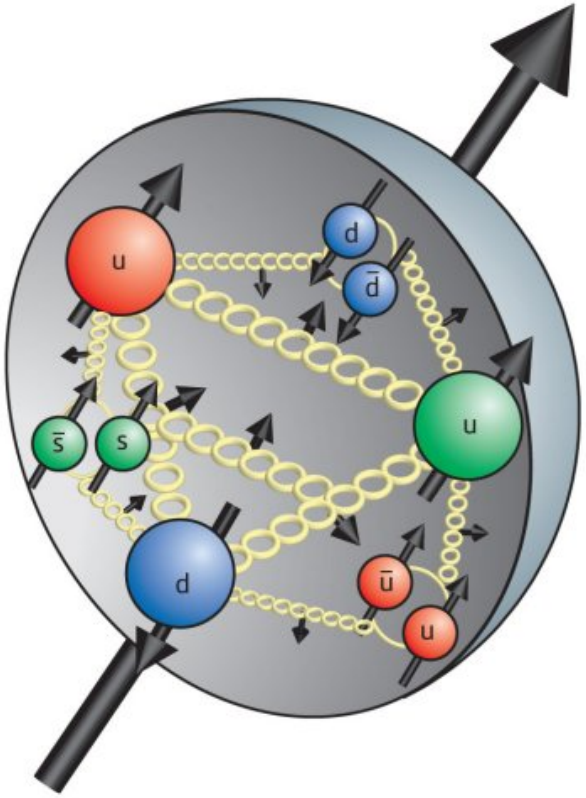
Outline

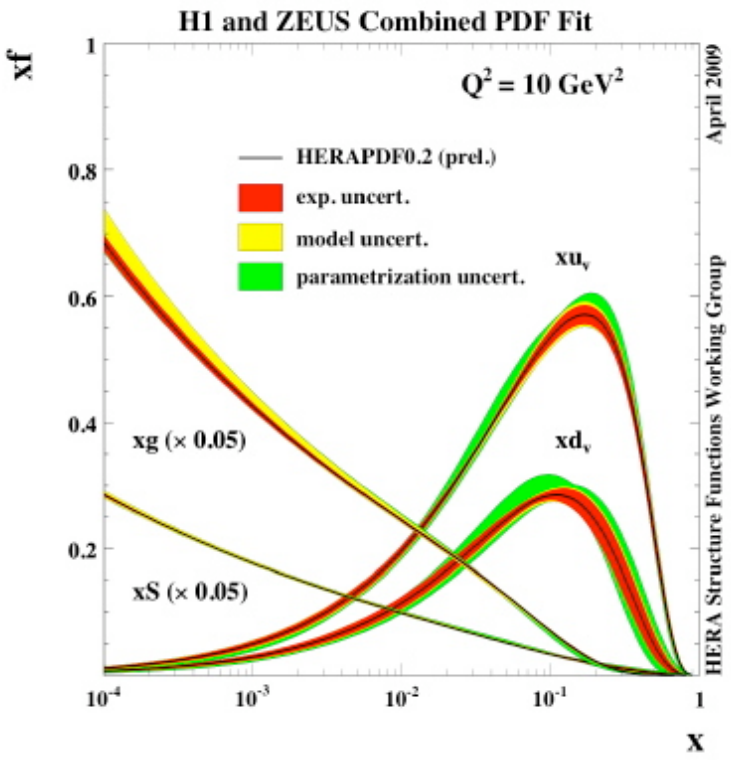
1. Forward Hadron Production in pA Collisions

2. NLO Forward Hadron Production in pA Collisions
   - The $q \to q$ channel
   - The other three channels: $g \to g$, $q \to g$ and $g \to q$

3. Conclusion and Outlook

Deep into low-x region of Protons

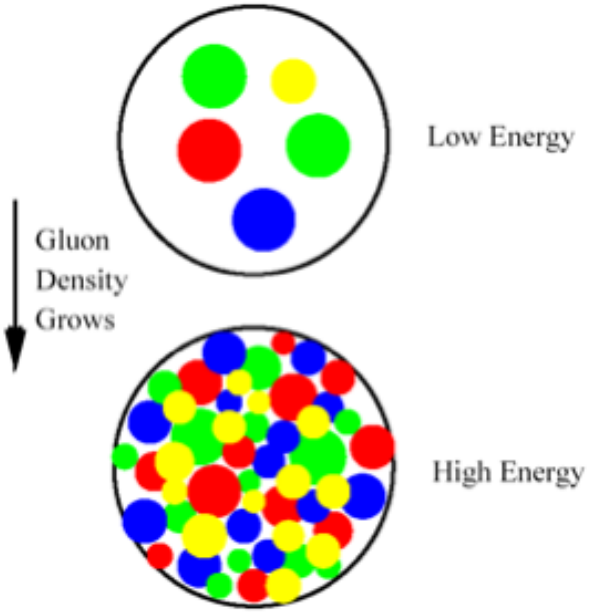




- Partons in the low-x region is dominated by gluons.
- Gluon splitting functions have $1/x$ singularities.
- Resummation of the $\alpha_s \ln \frac{1}{x}$.
- The dynamics becomes non-linear at high gluon density.

Saturation physics

Saturation physics describes the high density parton distributions in the high energy limit.



In QCD, the McLerran-Venugopalan Model describes high density gluon distribution in a relativistic large nucleus ($A \gg 1$) by solving the classical Yang-Mills equation:

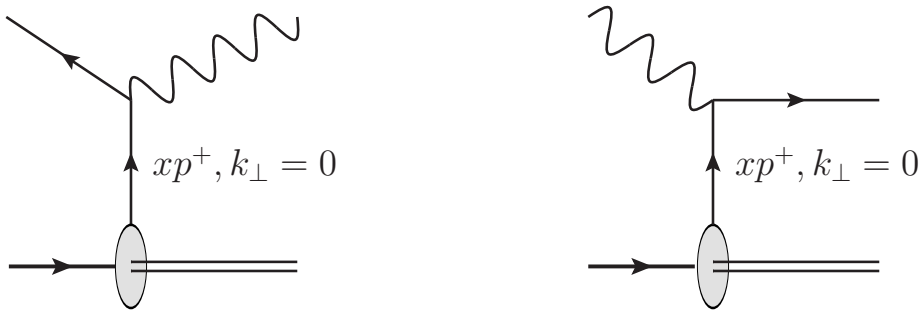
$$[D_\mu, F^{\mu\nu}] = gJ^\nu \quad \text{with} \quad J^\nu = \delta^{\nu+}\rho_a(x^-, x_\perp)T^a, \quad \text{COV gauge} \Rightarrow -\nabla_\perp^2 A^+ = g\rho.$$

The Wilson line

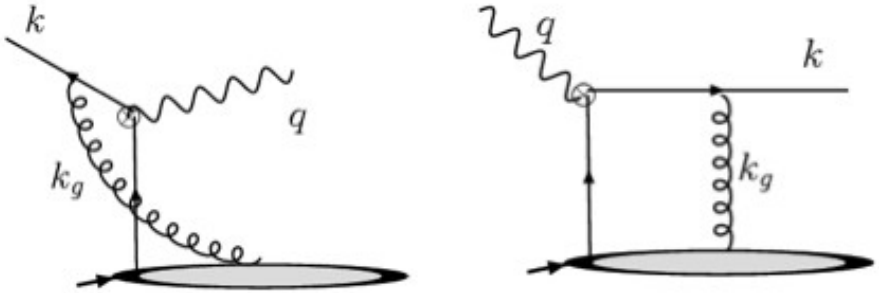
$$U(x_\perp) = \mathrm{T}\exp\left[-ig^2 \int \mathrm{d}z^- \mathrm{d}^2 z_\perp G\left(x_\perp - z_\perp\right) \rho\left(z^-, z_\perp\right)\right]$$

Collinear Factorization vs $k_\perp$ Factorization

Collinear Factorization Approximation: neglecting the $k_\perp$

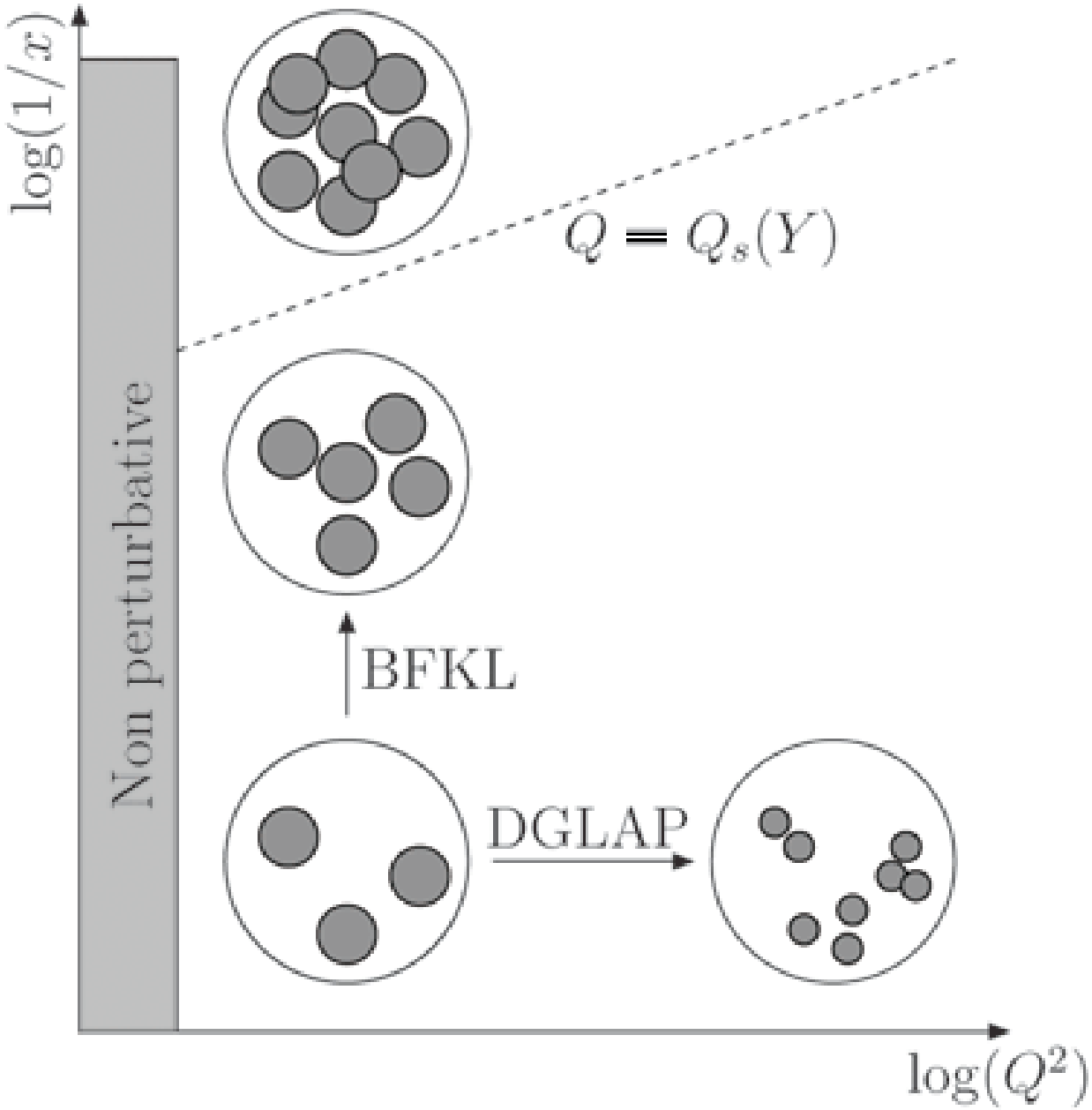


$k_\perp$ Factorization(Spin physics and saturation physics)



- In general, there is intrinsic $k_\perp \simeq \lambda$. It can be negligible for partons in protons, but should be taken into account for nucleus targets with $A \to \infty$.
- The accumulation of $k_\perp^2$ is proportional to $A^{\frac{1}{3}}\lambda^2$. (Random Walk Picture)
- $k_\perp$ Factorization: High energy evolution respect to x with $k_\perp$ unintegrated.
- Collinear Factorization: DGLAP evolution respect to both x and $k_\perp$.

Phase diagram in QCD



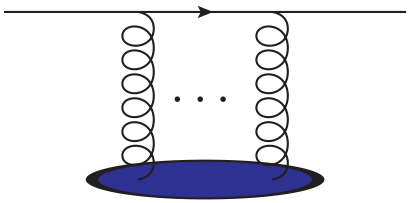
Forward hadron production in pA collisions

Consider the inclusive production of inclusive forward hadrons in pA collisions, i.e., in the process: [Dumitru, Jalilian-Marian, 02]

$$p + A \to H + X.$$

The leading order result for producing a hadron with transverse momentum $p_\perp$ at rapidity y_h

$$\frac{d\sigma_{\rm LO}^{pA\to hX}}{d^2p_\perp dy_h} = \int_\tau^1 \frac{dz}{z^2}\left[\sum_f x_p q_f(x_p)\mathcal{F}(k_\perp)D_{h/q}(z) + x_p g(x_p)\tilde{\mathcal{F}}(k_\perp)D_{h/g}(z)\right].$$



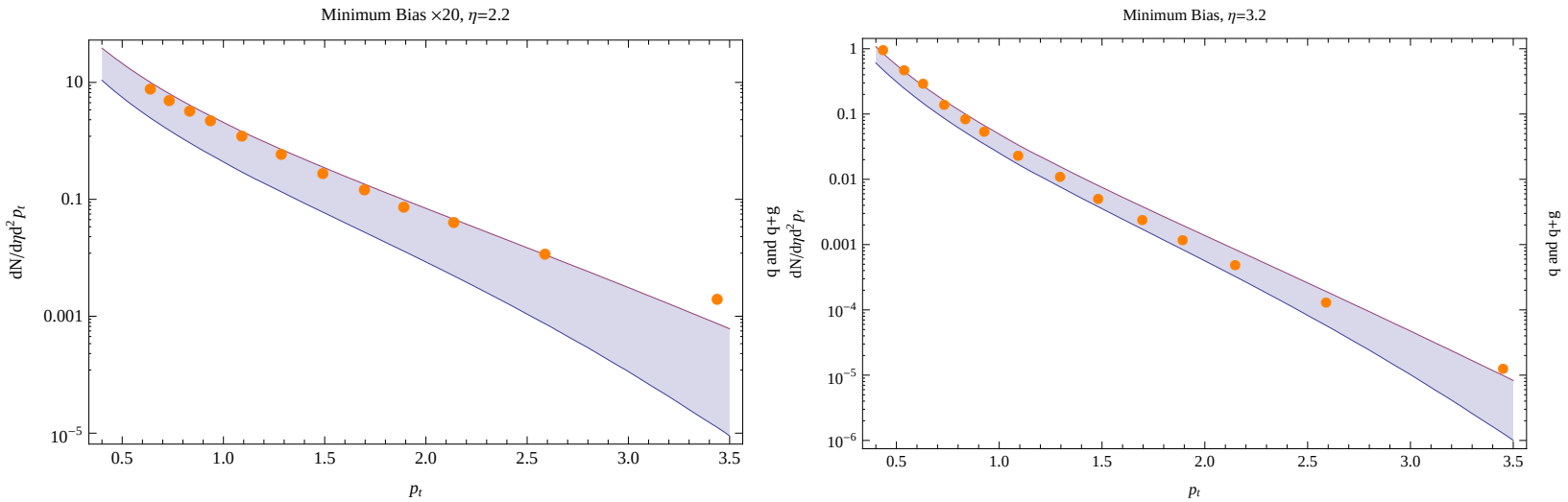
$$\Rightarrow \qquad U(x_\perp) = \mathcal{P}\exp\left\{ig_S\int_{-\infty}^{+\infty} \mathrm{d}x^+\, T^c A_c^-(x^+, x_\perp)\right\},$$

$$\mathcal{F}(k_\perp) = \int \frac{d^2x_\perp d^2y_\perp}{(2\pi)^2} e^{-ik_\perp\cdot(x_\perp - y_\perp)} S_Y^{(2)}(x_\perp, y_\perp).$$

- $p_\perp = zk_\perp$, $x_p = \frac{p_\perp}{z\sqrt{s}}e^{y_h}$ (large), $\tau = zx_p$ and $x_g = \frac{p_\perp}{z\sqrt{s}}e^{-y_h}$(small).
- $S_Y^{(2)}(x_\perp, y_\perp) = \frac{1}{N_c}\left\langle \mathrm{Tr}U(x_\perp)U^\dagger(y_\perp)\right\rangle_Y$ with $Y \sim \ln 1/x_g$.
- The gluon channel with $\tilde{\mathcal{F}}(k_\perp)$ defined in the adjoint representation.
- Beyond the hybrid factorization, see [E. Avsar, arXiv:1203.1916].

Issues with the leading order calculation

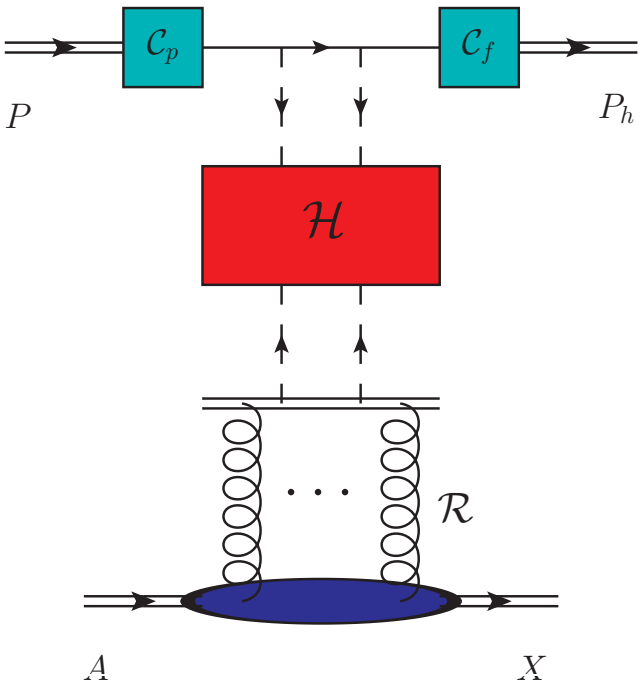
The comparison between the leading order calculation and the RHIC data:



Comments: Why do we need NLO calculations?

- LO calculation is order of magnitude estimate. Normally, we need to introduce the artificial K factor to fix the normalization. Fails to describe large $p_\perp$ data.
- There are large theoretical uncertainties due to renormalization/factorization scale dependence in $xf(x)$ and $D(z)$. NLO reduces the scale dependence. Higher order in the perturbative series in α_s improves the reliability of the predictions.
- $K = \frac{\sigma_{\rm LO}+\sigma_{\rm NLO}}{\sigma_{\rm LO}}$ is not a good approximation.
  NLO results may distort the shape of the cross section.
- NLO is vital in terms of establishing the QCD factorization in saturation physics.

The overall picture



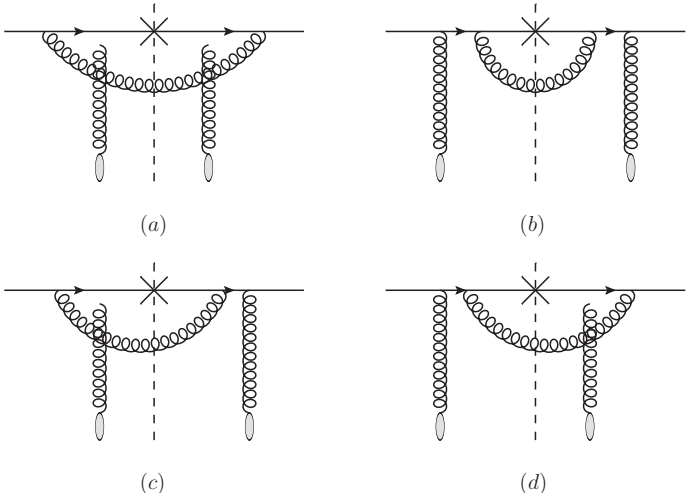
The QCD factorization formalism for this process reads as,

$$\frac{d^3\sigma^{p+A\to h+X}}{dyd^2p_\perp} = \sum_a \int \frac{dz}{z^2}\frac{dx}{x}\xi x f_a(x,\mu) D_{h/c}(z,\mu) \int [dx_\perp] S^Y_{a,c}([x_\perp]) \mathcal{H}_{a\to c}(\alpha_s,\xi,[x_\perp]\mu) \ .$$

- There is no rapidity divergence at LO. Encounter three types of divergences at NLO.
- For UGD, the rapidity divergence cannot be canceled between real final state and virtual gluon emission due to different restrictions on $k_\perp$.
- Subtractions of the divergences via renormalization.
  $\Rightarrow$ Finite results for hard factors at NLO.

The real contributions in the coordinate space

Computing the real diagrams with a quark ($b_\perp$) and a gluon ($x_\perp$) in the final state in the dipole model in the coordinate space: [G. Chirilli, BX and F. Yuan, 11;12]



$$\frac{d\sigma^{qA\to qgX}}{d^3k_1 d^3k_2} = \alpha_S C_F \delta(p^+ - k_1^+ - k_2^+) \int \frac{\mathrm{d}^2x_\perp}{(2\pi)^2}\frac{\mathrm{d}^2x'_\perp}{(2\pi)^2}\frac{\mathrm{d}^2b_\perp}{(2\pi)^2}\frac{\mathrm{d}^2b'_\perp}{(2\pi)^2}$$
$$\times e^{-ik_{1\perp}\cdot(x_\perp - x'_\perp)} e^{-ik_{2\perp}\cdot(b_\perp - b'_\perp)} \sum_{\lambda\alpha\beta} \psi^{\lambda *}_{\alpha\beta}(u'_\perp)\psi^{\lambda}_{\alpha\beta}(u_\perp)$$
$$\times \Big[S_Y^{(6)}(b_\perp, x_\perp, b'_\perp, x'_\perp) + S_Y^{(2)}(v_\perp, v'_\perp)$$
$$- S_Y^{(3)}(b_\perp, x_\perp, v'_\perp) - S_Y^{(3)}(v_\perp, x'_\perp, b'_\perp) \Big],$$

with $u_\perp = x_\perp - b_\perp$ and $v_\perp = (1-\xi)x_\perp + \xi b_\perp$.

The real contributions in the coordinate space

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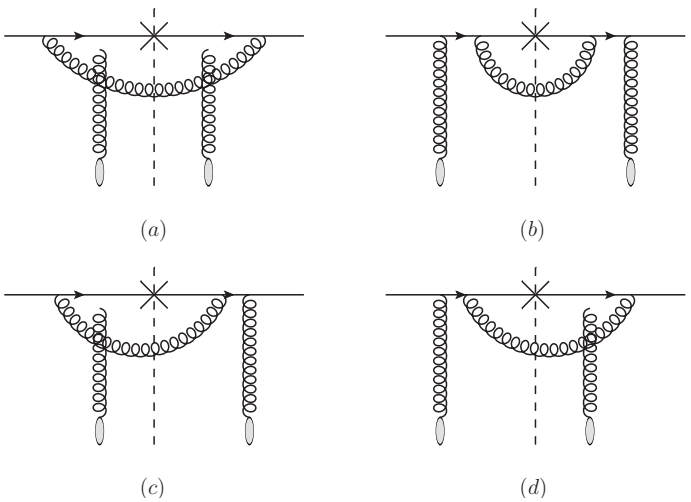


$$S_Y^{(6)}(b_\perp, x_\perp, b'_\perp, x'_\perp) = \frac{1}{C_F N_c} \left\langle \mathrm{Tr}\left(U(b_\perp)U^\dagger(b'_\perp)T^d T^c\right)\left[W(x_\perp)W^\dagger(x'_\perp)\right]^{cd} \right\rangle_Y ,$$

$$S_Y^{(3)}(b_\perp, x_\perp, v'_\perp) = \frac{1}{C_F N_c} \left\langle \mathrm{Tr}\left(U(b_\perp)T^d U^\dagger(v'_\perp)T^c\right) W^{cd}(x_\perp) \right\rangle_Y .$$

- By integrating over the gluon momentum, we identify $x_\perp$ to $x'_\perp$ which simplifies $S_Y^{(6)}(b_\perp, x_\perp, b'_\perp, x'_\perp)$ to $S^{(2)}(b_\perp, b'_\perp)$.
- $S_Y^{(3)}(b_\perp, x_\perp, v'_\perp) = \frac{N_c}{2C_F}\left[S_Y^{(4)}(b_\perp, x_\perp, v'_\perp) - \frac{1}{N_c^2} S_Y^{(2)}(b_\perp, v'_\perp)\right]$

The real contributions in the momentum space

By integrating over the gluon $(k_1^+, k_{1\perp})$, we can cast the real contribution into

$$\frac{\alpha_s}{2\pi^2}\int \frac{dz}{z^2} D_{h/q}(z) \int_{\tau/z}^{1} d\xi \frac{1+\xi^2}{1-\xi} xq(x) \left\{ C_F \int d^2k_{g\perp} \mathcal{I}(k_\perp, k_{g\perp}) + \frac{N_c}{2}\int d^2k_{g\perp} d^2k_{g1\perp} \mathcal{J}(k_\perp, k_{g\perp}, k_{g1\perp}) \right\},$$

where $x = \tau/z\xi$ and $\mathcal{I}$ and $\mathcal{J}$ are defined as

$$\mathcal{I}(k_\perp, k_{g\perp}) = \mathcal{F}(k_{g\perp}) \left[\frac{k_\perp - k_{g\perp}}{(k_\perp - k_{g\perp})^2} - \frac{k_\perp - \xi k_{g\perp}}{(k_\perp - \xi k_{g\perp})^2} \right]^2,$$

$$\mathcal{J}(k_\perp, k_{g\perp}, k_{g1\perp}) = \left[\mathcal{F}(k_{g\perp}) \delta^{(2)}(k_{g1\perp} - k_{g\perp}) - \mathcal{G}(k_{g\perp}, k_{g1\perp}) \right] \frac{2(k_\perp - \xi k_{g\perp}) \cdot (k_\perp - k_{g1\perp})}{(k_\perp - \xi k_{g\perp})^2 (k_\perp - k_{g1\perp})^2}$$

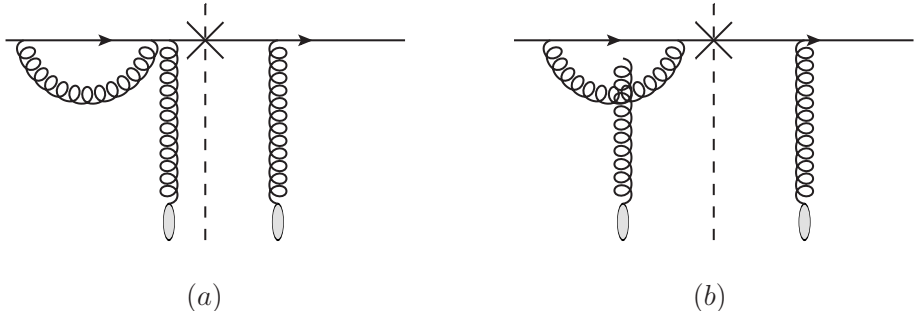
$$\text{with} \quad \mathcal{G}(k_\perp, l_\perp) = \int \frac{d^2x_\perp d^2y_\perp d^2b_\perp}{(2\pi)^4} e^{-ik_\perp \cdot (x_\perp - b_\perp) - il_\perp \cdot (b_\perp - y_\perp)} S_Y^{(4)}(x_\perp, b_\perp, y_\perp).$$

Three types of divergences: Life is boring without divergences.

- $\xi \to 1 \Rightarrow$ Rapidity divergence.
- $k_{g\perp} \to k_\perp \Rightarrow$ Collinear divergence associated with parton distributions.
- $k_{g\perp} \to k_\perp/\xi \Rightarrow$ Collinear divergence associated with fragmentation functions.

The virtual contributions in the momentum space

Now consider the virtual contribution



$$-2\alpha_s C_F \int \frac{d^2 v_\perp}{(2\pi)^2} \frac{d^2 v'_\perp}{(2\pi)^2} \frac{d^2 u_\perp}{(2\pi)^2} e^{-ik_\perp \cdot (v_\perp - v'_\perp)} \sum_{\lambda\alpha\beta} \psi^{\lambda *}_{\alpha\beta}(u_\perp) \psi^{\lambda}_{\alpha\beta}(u_\perp)$$

$$\times \left[S_Y^{(2)}(v_\perp, v'_\perp) - S_Y^{(3)}(b_\perp, x_\perp, v'_\perp) \right]$$

$$\Rightarrow \quad -\frac{\alpha_s}{2\pi^2} \int \frac{dz}{z^2} D_{h/q}(z) x_p q(x_p) \int_0^1 d\xi \frac{1+\xi^2}{1-\xi}$$

$$\times \left\{ C_F \int d^2 q_\perp \mathcal{I}(q_\perp, k_\perp) + \frac{N_c}{2} \int d^2 q_\perp d^2 k_{g1\perp} \mathcal{J}(q_\perp, k_\perp, k_{g1\perp}) \right\}.$$

Three types of divergences:

- $\xi \to 1 \Rightarrow$ Rapidity divergence.
- Collinear divergence associated with parton distributions and fragmentation functions.

The subtraction of the rapidity divergence

We remove the rapidity divergence from the real and virtual diagrams by the following subtraction:

$$\begin{aligned}\mathcal{F}(k_\perp) &= \mathcal{F}^{(0)}(k_\perp) - \frac{\alpha_s N_c}{2\pi^2}\int_0^1 \frac{d\xi}{1-\xi}\int \frac{d^2x_\perp d^2y_\perp d^2b_\perp}{(2\pi)^2} e^{-ik_\perp\cdot(x_\perp - y_\perp)} \\ &\quad \times \frac{(x_\perp - y_\perp)^2}{(x_\perp - b_\perp)^2(y_\perp - b_\perp)^2}\left[S^{(2)}(x_\perp, y_\perp) - S^{(4)}(x_\perp, b_\perp, y_\perp)\right].\end{aligned}$$

Decomposing the dipole splitting kernel as

$$\frac{(x_\perp - y_\perp)^2}{(x_\perp - b_\perp)^2(y_\perp - b_\perp)^2} = \frac{1}{(x_\perp - b_\perp)^2} + \frac{1}{(y_\perp - b_\perp)^2} - \frac{2(x_\perp - b_\perp)\cdot(y_\perp - b_\perp)}{(x_\perp - b_\perp)^2(y_\perp - b_\perp)^2}.$$

with the first two terms removed from the virtual diagrams while the last term removed from the real diagrams. Comments:

- This divergence removing procedure is similar to the renormalization of parton distribution and fragmentation function in collinear factorization.
- Splitting functions becomes $\frac{1+\xi^2}{(1-\xi)_+}$ after the subtraction.
- Rapidity divergence disappears when the $k_\perp$ is integrated. Unique feature of unintegrated gluon distributions.

The subtraction of the rapidity divergence

$$\mathcal{F}(k_\perp) = \mathcal{F}^{(0)}(k_\perp) - \frac{\alpha_s N_c}{2\pi^2}\int_0^1 \frac{d\xi}{1-\xi}\int \frac{d^2x_\perp d^2y_\perp d^2b_\perp}{(2\pi)^2} e^{-ik_\perp\cdot(x_\perp - y_\perp)} \times \frac{(x_\perp - y_\perp)^2}{(x_\perp - b_\perp)^2(y_\perp - b_\perp)^2}\left[S^{(2)}(x_\perp, y_\perp) - S^{(4)}(x_\perp, b_\perp, y_\perp)\right].$$

This is equivalent to the Balitsky-Kovchegov equation:

$$\frac{\partial}{\partial Y} S_Y^{(2)}(x_\perp, y_\perp) = -\frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2b_\perp\,(x_\perp - y_\perp)^2}{(x_\perp - b_\perp)^2(y_\perp - b_\perp)^2}\left[S_Y^{(2)}(x_\perp, y_\perp) - S_Y^{(4)}(x_\perp, b_\perp, y_\perp)\right].$$

- Recall that $\mathcal{F}(k_\perp) = \int \frac{d^2x_\perp d^2y_\perp}{(2\pi)^2} e^{-ik_\perp\cdot(x_\perp - y_\perp)} S^{(2)}(x_\perp, y_\perp)$.
- The rapidity divergence is an artifact of the high energy limit $s \to \infty$. By slightly tilting away from the light cone, we obtain $\int_0^1 \frac{d\xi}{1-\xi+e^{-Y}}$ and introduce Y dependence to the dipole amplitude.
- Introducing cutoff $1 - e^{-Y}$ and considering the change from Y to $Y + dY \Rightarrow$ the BK equation
- $\xi \to 1$ implies that gluon is infinitely soft and its rapidity goes to $-\infty$. This soft gluon is in fact collinear to the target nucleus.
- Renormalize the soft gluon into the gluon distribution function of the target nucleus through the BK evolution equation.

The subtraction of the collinear divergence

Let us take the following integral as an example:

$$
\begin{aligned}
I_1(k_\perp) &= \int \frac{d^2 k_{g\perp}}{(2\pi)^2} \mathcal{F}(k_{g\perp}) \frac{1}{(k_\perp - k_{g\perp})^2} \,, \\
&= \frac{1}{4\pi} \int \frac{d^2 x_\perp d^2 y_\perp}{(2\pi)^2} e^{-ik_\perp \cdot r_\perp} S_Y^{(2)}(x_\perp, y_\perp) \left(-\frac{1}{\hat{\epsilon}} + \ln \frac{c_0^2}{\mu^2 r_\perp^2} \right) ,
\end{aligned}
$$

where $c_0 = 2e^{-\gamma_E}$, γ_E is the Euler constant and $r_\perp = x_\perp - y_\perp$.

- Use dimensional regularization ($D = 4 - 2\epsilon$) and the $\overline{\text{MS}}$ subtraction scheme ($\frac{1}{\hat{\epsilon}} = \frac{1}{\epsilon} - \gamma_E + \ln 4\pi$).
- $\int \frac{d^2 k_{g\perp}}{(2\pi)^2} \Rightarrow \mu^{2\epsilon} \int \frac{d^{2-2\epsilon} k_{g\perp}}{(2\pi)^{2-2\epsilon}}$ where μ is the renormalization scale dependence coming from the strong coupling g.
- The terms proportional to the collinear divergence $\frac{1}{\hat{\epsilon}}$ should be factorized either into parton distribution functions or fragmentation functions.

The subtraction of the collinear divergence

Remove the collinear singularities by redefining the quark distribution and the quark fragmentation function as follows

$$
\begin{aligned}
q(x,\mu) &= q^{(0)}(x) - \frac{1}{\hat{\epsilon}}\frac{\alpha_s(\mu)}{2\pi}\int_x^1 \frac{d\xi}{\xi} C_F \mathcal{P}_{qq}(\xi) q\left(\frac{x}{\xi}\right), \\
D_{h/q}(z,\mu) &= D^{(0)}_{h/q}(z) - \frac{1}{\hat{\epsilon}}\frac{\alpha_s(\mu)}{2\pi}\int_z^1 \frac{d\xi}{\xi} C_F \mathcal{P}_{qq}(\xi) D_{h/q}\left(\frac{z}{\xi}\right),
\end{aligned}
$$

with

$$
\mathcal{P}_{qq}(\xi) = \underbrace{\frac{1+\xi^2}{(1-\xi)_+}}_{\text{Real Sub}} + \underbrace{\frac{3}{2}\delta(1-\xi)}_{\text{Virtual Sub}}.
$$

Comments:

- Reproducing the DGLAP equation for the quark channel. Other channels will complete the full equation.
- The emitted gluon is collinear to the initial state quark $\Rightarrow$ Renormalization of the parton distribution.
- The emitted gluon is collinear to the final state quark $\Rightarrow$ Renormalization of the fragmentation function.

Hard Factors

For the $q \to q$ channel, the factorization formula can be written as

$$\frac{d^3\sigma^{p+A\to h+X}}{dyd^2p_\perp} = \int \frac{dz}{z^2}\frac{dx}{x}\xi x q(x,\mu) D_{h/q}(z,\mu) \int \frac{d^2x_\perp d^2y_\perp}{(2\pi)^2}\left\{S_Y^{(2)}(x_\perp,y_\perp)\left[\mathcal{H}_{2qq}^{(0)} + \frac{\alpha_s}{2\pi}\mathcal{H}_{2qq}^{(1)}\right]\right.$$
$$\left. + \int \frac{d^2b_\perp}{(2\pi)^2} S_Y^{(4)}(x_\perp,b_\perp,y_\perp)\frac{\alpha_s}{2\pi}\mathcal{H}_{4qq}^{(1)}\right\}$$

with $\mathcal{H}_{2qq}^{(0)} = e^{-ik_\perp\cdot r_\perp}\delta(1-\xi)$ and

$$\mathcal{H}_{2qq}^{(1)} = C_F\mathcal{P}_{qq}(\xi)\ln\frac{c_0^2}{r_\perp^2\mu^2}\left(e^{-ik_\perp\cdot r_\perp} + \frac{1}{\xi^2}e^{-i\frac{k_\perp}{\xi}\cdot r_\perp}\right) - 3C_F\delta(1-\xi)e^{-ik_\perp\cdot r_\perp}\ln\frac{c_0^2}{r_\perp^2k_\perp^2}$$
$$-(2C_F - N_c)\,e^{-ik_\perp\cdot r_\perp}\left[\frac{1+\xi^2}{(1-\xi)_+}\widetilde{I}_{21} - \left(\frac{(1+\xi^2)\ln(1-\xi)^2}{1-\xi}\right)_+\right]$$

$$\mathcal{H}_{4qq}^{(1)} = -4\pi N_c e^{-ik_\perp\cdot r_\perp}\left\{e^{-i\frac{1-\xi}{\xi}k_\perp\cdot(x_\perp-b_\perp)}\frac{1+\xi^2}{(1-\xi)_+}\frac{1}{\xi}\frac{x_\perp-b_\perp}{(x_\perp-b_\perp)^2}\cdot\frac{y_\perp-b_\perp}{(y_\perp-b_\perp)^2}\right.$$
$$-\delta(1-\xi)\int_0^1 d\xi'\frac{1+\xi'^2}{(1-\xi')_+}\left[\frac{e^{-i(1-\xi')k_\perp\cdot(y_\perp-b_\perp)}}{(b_\perp-y_\perp)^2} - \delta^{(2)}(b_\perp-y_\perp)\int d^2r'_\perp\frac{e^{ik_\perp\cdot r'_\perp}}{r'^2_\perp}\right]$$

where $$\widetilde{I}_{21} = \int\frac{d^2b_\perp}{\pi}\left\{e^{-i(1-\xi)k_\perp\cdot b_\perp}\left[\frac{b_\perp\cdot(\xi b_\perp - r_\perp)}{b_\perp^2(\xi b_\perp - r_\perp)^2} - \frac{1}{b_\perp^2}\right] + e^{-ik_\perp\cdot b_\perp}\frac{1}{b_\perp^2}\right\}.$$

Hard Factors in the MV model

In the MV model with $S^{(2)}_{\rm MV}(x_\perp, y_\perp) = \exp\left[-\frac{(x_\perp - y_\perp)^2 Q_s^2}{4}\right]$, the factorization formula can be written as

$$\frac{d^3\sigma^{p+A\to h+X}}{dyd^2p_\perp} = \int \frac{dz}{z^2}\frac{dx}{x}\xi x q(x,\mu) D_{h/q}(z,\mu)\left[\bar{\mathcal{H}}^{(0)}_{2qq} + \frac{\alpha_s}{2\pi}\bar{\mathcal{H}}^{(1)}_{2qq} + \frac{\alpha_s}{2\pi}\bar{\mathcal{H}}^{(1)}_{4qq}\right]$$

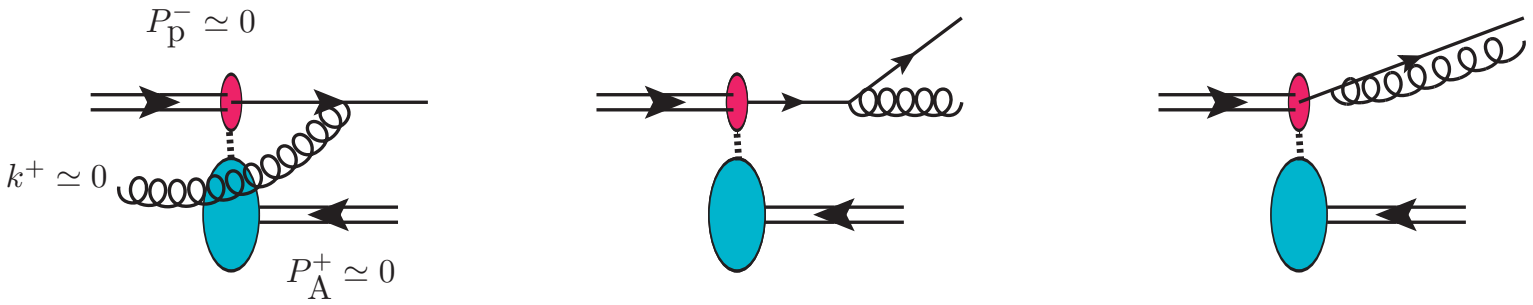
with $\bar{\mathcal{H}}^{(0)}_{2qq} = \delta(1-\xi)\mathcal{F}(k_\perp)$ and

$$\begin{aligned}
\bar{\mathcal{H}}^{(1)}_{2qq} \quad = \quad & \frac{N_c}{2}\mathcal{P}_{qq}(\xi)\mathcal{F}(k_\perp)\left[\ln\frac{Q_s^2}{\mu^2 e^{\gamma_E}} + \exp\left(\frac{k_\perp^2}{Q_s^2}\right)L^{(1,0)}\left(-1,-\frac{k_\perp^2}{Q_s^2}\right)\right] \\
& + \frac{1}{\xi^2}\frac{N_c}{2}\mathcal{P}_{qq}(\xi)\mathcal{F}\left(\frac{k_\perp}{\xi}\right)\left[\ln\frac{Q_s^2}{\mu^2 e^{\gamma_E}} + \exp\left(\frac{k_\perp^2}{\xi^2 Q_s^2}\right)L^{(1,0)}\left(-1,-\frac{k_\perp^2}{\xi^2 Q_s^2}\right)\right] \\
& - \delta(1-\xi)\frac{3N_c}{2}\mathcal{F}(k_\perp)\left[\ln\frac{Q_s^2}{k_\perp^2 e^{\gamma_E}} + \exp\left(\frac{k_\perp^2}{Q_s^2}\right)L^{(1,0)}\left(-1,-\frac{k_\perp^2}{Q_s^2}\right)\right], \\
\bar{\mathcal{H}}^{(1)}_{4qq} \quad = \quad & N_c\delta(1-\xi)\mathcal{F}(k_\perp)\left[\frac{3}{2}\ln\frac{Q_s^2}{k_\perp^2 e^{\gamma_E}} + \int_0^1 d\xi'\,\frac{1+\xi'^2}{(1-\xi')_+}\exp\left(-\frac{\xi'^2 k_\perp^2}{Q_s^2}\right)L^{(1,0)}\left(-1,\frac{\xi'^2 k_\perp^2}{Q_s^2}\right)\right] \\
& - \frac{S_\perp N_c}{\pi}\frac{1+\xi^2}{(1-\xi)_+}\frac{1}{k_\perp^2}\left[1-\exp\left(-\frac{k_\perp^2}{Q_s^2}\right)\right]\left[1-\exp\left(-\frac{k_\perp^2}{\xi^2 Q_s^2}\right)\right]
\end{aligned}$$

- Large N_c limit $\Rightarrow$ factorization of $S^{(4)}_Y(x_\perp, b_\perp, y_\perp) \to S^{(2)}_Y(x_\perp, b_\perp)S^{(2)}_Y(b_\perp, y_\perp)$.

What have we learnt so far?

- Achieve a systematic factorization for the $p + A \to H + X$ process.
- Gluons in different kinematical region give different divergences. 1.soft, collinear to the target nucleus; 2. collinear to the initial quark; 3. collinear to the final quark.

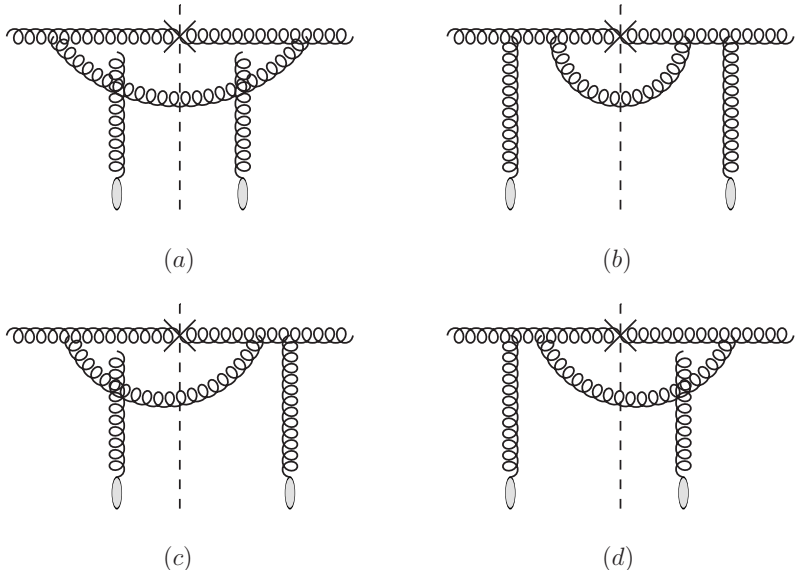


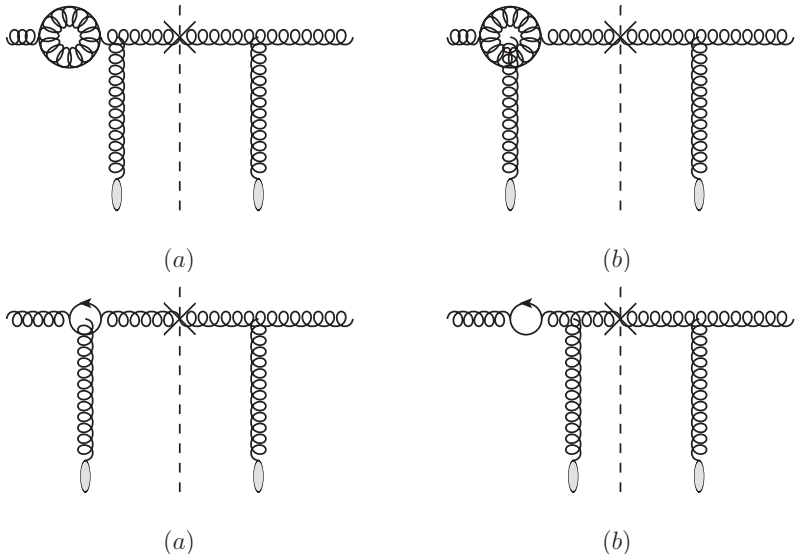
Rapidity Divergence | Collinear Divergence (P) | Collinear Divergence (F)

- Factorization scale μ can be set to $c_0/r_\perp \simeq Q_s$.
- Large N_c limit simplifies the calculation quite a lot.
- Consistent check: take the dilute limit, $k_\perp^2 \gg Q_s^2$, the result is consistent with the leading order collinear factorization formula. Good large $p_\perp$ behavior!
- The NLO prediction and test of saturation physics now is not only conceivable but also practicable!
- More comments at the end on the implementation of the phenomenological applications.

The $g \to g$ channel

The calculation on the $g \to g$ channel is similar to the $q \to q$ channel with a few differences:





- Two types of virtual graphs.
- The subtraction of the rapidity divergence follows

$$\frac{\partial}{\partial Y}\left\langle \mathrm{tr}_A W^\dagger_{x_\perp} W_{y_\perp}\right\rangle_Y = -\frac{\alpha_s}{\pi^2}\int \frac{d^2 z_\perp\,(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2 (y_\perp - z_\perp)^2} \left[C_A \left\langle \mathrm{tr}_A W^\dagger_{x_\perp} W_{y_\perp}\right\rangle_Y - \left\langle \mathrm{tr}_A W^\dagger_{z_\perp} t^a W_{z_\perp} W^\dagger_{x_\perp} t^a W_{y_\perp}\right\rangle_Y\right] ,$$

  where W is the Wilson line in the adjoint representation.
- $W^{ab}(x_\perp) = 2\mathrm{Tr}\left[T^a U(x_\perp) T^b U^\dagger(x_\perp)\right]$ converts everything into Fundamental representation.
- Large N_c limit is essential here to eliminate the sextupole (6 U's in a single trace) contribution.

The subtraction of the collinear divergence

Remove the collinear singularities by redefining the parton distribution and the fragmentation function as follows

$$\begin{bmatrix} q(x,\mu) \\ g(x,\mu) \end{bmatrix} = \begin{bmatrix} q^{(0)}(x) \\ g^{(0)}(x) \end{bmatrix} - \frac{1}{\hat{\epsilon}} \frac{\alpha(\mu)}{2\pi} \int_x^1 \frac{d\xi}{\xi} \begin{bmatrix} C_F P_{qq}(\xi) & T_R P_{qg}(\xi) \\ C_F P_{gq}(\xi) & N_c P_{gg}(\xi) \end{bmatrix} \begin{bmatrix} q(x/\xi) \\ g(x/\xi) \end{bmatrix},$$

and

$$\begin{bmatrix} D_{h/q}(z,\mu) \\ D_{h/g}(z,\mu) \end{bmatrix} = \begin{bmatrix} D^{(0)}_{h/q}(z) \\ D^{(0)}_{h/g}(z) \end{bmatrix} - \frac{1}{\hat{\epsilon}} \frac{\alpha(\mu)}{2\pi} \int_z^1 \frac{d\xi}{\xi} \begin{bmatrix} C_F P_{qq}(\xi) & C_F P_{gq}(\xi) \\ T_R P_{qg}(\xi) & N_c P_{gg}(\xi) \end{bmatrix} \begin{bmatrix} D_{h/q}(z/\xi) \\ D_{h/g}(z/\xi) \end{bmatrix},$$

with

$$\mathcal{P}_{gg}(\xi) = \underbrace{2\left[\frac{\xi}{(1-\xi)_+} + \frac{1-\xi}{\xi} + \xi(1-\xi)\right]}_{\text{Real}} + \underbrace{\left(\frac{11}{6} - \frac{2N_f T_R}{3N_c}\right)\delta(1-\xi)}_{\text{Virtual}},$$

$$\mathcal{P}_{gq}(\xi) = \frac{1}{\xi}\left[1 + (1-\xi)^2\right] \qquad \mathcal{P}_{qg}(\xi) = \left[(1-\xi)^2 + \xi^2\right]$$

Comments:

- Reproducing the full DGLAP equation for the quark channel.
- $q \to g$ and $g \to q$ channels only have collinear divergences, no rapidity divergence.

The LO+NLO cross section

With chosen $\mu = c_0/r_\perp$, adding all the channels together in the large N_c limit gives

$$\frac{d^3\sigma^{p+A\to h+X}}{dyd^2p_\perp} = \int \frac{dz}{z^2}\frac{dx}{x}\xi\left[xq(x,\mu), xg(x,\mu)\right]\begin{bmatrix} S_{qq} & S_{qg} \\ S_{gq} & S_{gg}\end{bmatrix}\begin{bmatrix} D_{h/q}(z,\mu) \\ D_{h/g}(z,\mu)\end{bmatrix},$$

$$\begin{aligned}
S_{qq} &= \int \frac{d^2x_\perp d^2y_\perp e^{-ik_\perp\cdot r_\perp}\delta(1-\xi)}{(2\pi)^2} S_Y^{(2)}(x_\perp, y_\perp)\left[1 - \frac{\alpha_s}{2\pi}3C_F \ln\frac{c_0^2}{r_\perp^2 k_\perp^2}\right] \\
&\quad + \int \frac{d^2x_\perp d^2y_\perp d^2b_\perp}{(2\pi)^4} S_Y^{(4)}(x_\perp, b_\perp, y_\perp)\frac{\alpha_s}{2\pi}\mathcal{H}_{4qq}^{(1)}, \\
S_{qg} &= \frac{\alpha_s}{2\pi}\int \frac{d^2x_\perp d^2y_\perp d^2b_\perp}{(2\pi)^4} S_Y^{(4)}(x_\perp, b_\perp, y_\perp)\mathcal{H}_{4gq}^{(1)}, \\
S_{gq} &= \frac{\alpha_s}{2\pi}\int \frac{d^2x_\perp d^2y_\perp d^2b_\perp}{(2\pi)^4} S_Y^{(4)}(x_\perp, b_\perp, y_\perp)\mathcal{H}_{4qg}^{(1)}, \\
S_{gg} &= \int \frac{d^2x_\perp d^2y_\perp e^{-ik_\perp\cdot r_\perp}\delta(1-\xi)}{(2\pi)^2}\left|S_Y^{(2)}(x_\perp, y_\perp)\right|^2\left[1 - \frac{\alpha_s N_c}{2\pi}\left[\frac{11}{3} - \frac{4N_f T_R}{3N_c}\right]\ln\frac{c_0^2}{r_\perp^2 k_\perp^2}\right] \\
&\quad + \int \frac{d^2x_\perp d^2y_\perp d^2b_\perp}{(2\pi)^4} S_Y^{(2)}(x_\perp, b_\perp)S_Y^{(2)}(b_\perp, y_\perp)\frac{\alpha_s}{2\pi}\mathcal{H}_{2q\bar{q}}^{(1)} \\
&\quad + \int \frac{d^2x_\perp d^2y_\perp d^2b_\perp}{(2\pi)^4} S_Y^{(2)}(x_\perp, b_\perp)S_Y^{(2)}(b_\perp, y_\perp)S_Y^{(2)}(y_\perp, x_\perp)\frac{\alpha_s}{2\pi}\mathcal{H}_{6gg}^{(1)}.
\end{aligned}$$

The LO+NLO cross section continued

The hard factors are defined as

$$
\begin{aligned}
\mathcal{H}^{(1)}_{2q\bar{q}} &= 8\pi N_f T_R e^{-ik_\perp\cdot(y_\perp-b_\perp)}\delta(1-\xi)\int_0^1 d\xi'\left[\xi'^2+(1-\xi')^2\right] \\
&\quad\times\left[\frac{e^{-i\xi' k_\perp\cdot(x_\perp-y_\perp)}}{(x_\perp-y_\perp)^2}-\delta^{(2)}(x_\perp-y_\perp)\int d^2r'_\perp\frac{e^{ik_\perp\cdot r'_\perp}}{r'^2_\perp}\right] \\
\mathcal{H}^{(1)}_{6gg} &= -16\pi N_c e^{-ik_\perp\cdot r_\perp}\left\{e^{-i\frac{k_\perp}{\xi}\cdot(y-b)}\frac{[1-\xi(1-\xi)]^2}{(1-\xi)_+}\frac{1}{\xi^2}\frac{x_\perp-y_\perp}{(x_\perp-y_\perp)^2}\cdot\frac{b_\perp-y_\perp}{(b_\perp-y_\perp)^2}\right. \\
&\quad-\delta(1-\xi)\int_0^1 d\xi'\left[\frac{\xi'}{(1-\xi')_+}+\frac{1}{2}\xi'(1-\xi')\right] \\
&\quad\left.\times\left[\frac{e^{-i\xi' k_\perp\cdot(y_\perp-b_\perp)}}{(b_\perp-y_\perp)^2}-\delta^{(2)}(b_\perp-y_\perp)\int d^2r'_\perp\frac{e^{ik_\perp\cdot r'_\perp}}{r'^2_\perp}\right]\right\}, \\
\mathcal{H}^{(1)}_{4gq} &= -4\pi N_c\mathcal{W}\left(\frac{k_\perp}{\xi},k_\perp\right)P_{gq}(\xi)\frac{1}{\xi}\frac{x_\perp-y_\perp}{(x_\perp-y_\perp)^2}\cdot\frac{b_\perp-y_\perp}{(b_\perp-y_\perp)^2}, \\
\mathcal{H}^{(1)}_{4qg} &= -4\pi\mathcal{W}\left(k_\perp,\frac{k_\perp}{\xi}\right)P_{qg}(\xi)\frac{1}{\xi}\frac{x_\perp-y_\perp}{(x_\perp-y_\perp)^2}\cdot\frac{b_\perp-y_\perp}{(b_\perp-y_\perp)^2}, \\
&\quad\text{with}\quad \mathcal{W}(k_{1\perp},k_{2\perp}) = e^{-ik_{1\perp}\cdot(x_\perp-y_\perp)-ik_{2\perp}\cdot(y_\perp-b_\perp)}.
\end{aligned}
$$

Numerical implementation of the NLO result

Consistent implementation should include all the α_s corrections.

- NLO parton distributions. (Choose your favorite one, CTEQ or MSTW)
- NLO fragmentation function. (DSS or others.)
- Use NLO hard factors.
- Use the one-loop approximation for the running coupling which is sufficient in this calculation.
- NLO BK evolution equation for the dipole gluon distribution. (Hard)
  Alternate solution: Treat the dipole amplitude as an input, use GBW model or your favorite parametrization of dipole amplitudes with appropriate energy dependence, and then find the best fit by comparing with all the available data. Then make prediction for the LHC pA collisions. (working in progress)
- Looking at about $20 - 30$ percent uncertainty. Large N_c limit gives about 10 percent.

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Conclusion and Outlook

- We calculate inclusive hadron productions in pA collisions in the small-x saturation formalism at one-loop order.
- The rapidity divergence with small-x dipole gluon distribution of the nucleus is factorized into the BK evolution of the dipole gluon distribution function.
- The collinear divergences associated with the incoming parton distribution of the nucleon and the outgoing fragmentation function of the final state hadron are factorized into the well-known DGLAP equation.
- The hard coefficient function, which is finite and free of divergence of any kind, is evaluated at one-loop order.
- Now we have a systematic NLO description of inclusive forward hadron productions in pA collisions which is ready for making reliable predictions and conducting precision test. Phenomenological applications are promising for both RHIC and LHC experiments.

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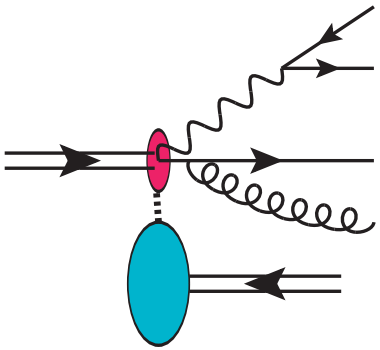
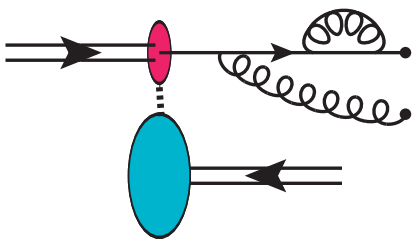
Outlook

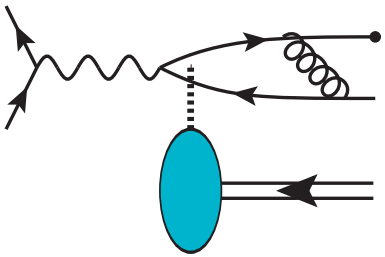
Using this factorization technique, we can imagine that a lot of other NLO calculations can be achieved in the near future.



NLO Drell-Yan productions



NLO dijet productions



NLO Single Inclusive DIS

- Drell-Yan lepton pair production in pA collisions at NLO. (work in process)
- NLO dijet productions in pA collisions. (Hard)
- Single inclusive DIS at NLO. (see similar work [Balitsky, Chirilli, 10], [Beuf, 11])
- Direct photon production in pA collisions at NLO (see [Jalilian-Marian, Rezaeian, 12]) and NNLO (similar to the DY case at NLO). Universality and large N_c
- Factorization beyond the hybrid formalism by using the $k_\perp$ dependent parton distributions (from the proton side) and fragmentation functions. (work in progress)